Perspective transform

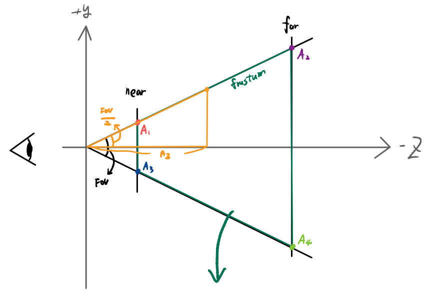
perspective (FOV , aspect , near , far) : M

aspect $\rightarrow \frac{\text{width}}{\text{height}}$

let $A = [A_x\ A_y\ A_z\ 1]$ as position input (a_position)

$P = [P_x\ P_y\ P_z\ P_w]$ as position output (gl_Position)

$P' = [P'_x\ P'_y\ P'_z] = [\frac{P_x}{P_w}\ \frac{P_y}{P_w}\ \frac{P_z}{P_w}]$



$P_w = A_z \cdot (-1)$ because A_z is on $-Z$

$\hookrightarrow M_w$

A_{1z} on $-Z$, A_{1y} is positive

$A_{1y} = -A_{1z} \cdot \tan(\frac{FOV}{2})$

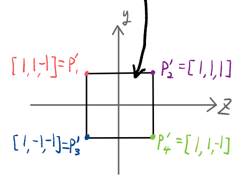
for $P_y = M_y \cdot A_y$, 求 M_y

we know $P'_{1y} = 1 = \frac{P_{1y}}{P_{1w}} = \frac{M_y \cdot A_{1y}}{P_{1w}} = \frac{M_y \cdot A_{1y}}{-A_{1z}} = \frac{M_y \cdot (\cancel{-A_{1z}} \cdot \tan(\frac{FOV}{2}))}{\cancel{-A_{1z}}}$

$\Rightarrow M_y = \frac{1}{\tan(\frac{FOV}{2})} = \tan(\frac{\pi}{2} - \frac{FOV}{2})$

$\Rightarrow P_y = A_y \cdot \tan(\frac{\pi}{2} - \frac{FOV}{2})$

apply M and divide by P_w



$A_{1x} = A_{1y} \cdot \frac{\text{width}}{\text{height}} = A_{1y} \cdot \text{aspect}$

for $P_x = M_x \cdot A_x$, 求 M_x,

we know $P'_{1x} = 1 = \frac{P_{1x}}{P_{1w}} = \frac{P_{1x}}{-A_{1z}} = \frac{M_x A_{1x}}{-A_{1z}} = \frac{M_x \cdot A_{1y} \cdot \text{aspect}}{-A_{1z}} = \frac{M_x \cdot \cancel{(-A_z)} \cdot \tan(\frac{FOV}{2}) \cdot \text{aspect}}{\cancel{-A_z}}$

$\Rightarrow M_x = \frac{1}{\tan(\frac{FOV}{2}) \cdot \text{aspect}} = \frac{\tan(\frac{\pi}{2} - \frac{FOV}{2})}{\text{aspect}}$

$\Rightarrow P_x = A_x \frac{\tan(\frac{\pi}{2} - \frac{FOV}{2})}{\text{aspect}}$

for $P_z = M_z \cdot A_z + M_{zt}$, 求 M_z M_{zt}

$\begin{cases} A_{1z} = -\text{near} \\ A_{2z} = -\text{far} \end{cases}$

we know $\begin{cases} P'_{1z} = -1 = \frac{P_z}{P_w} = \frac{M_z A_{1z} + M_{zt}}{-A_{1z}} = \frac{-\text{near} \cdot M_z + M_{zt}}{\text{near}} \\ P'_{2z} = 1 = \frac{P_z}{P_w} = \frac{M_z A_{2z} + M_{zt}}{-A_{2z}} = \frac{-\text{far}\ M_z + M_{zt}}{\text{far}} \end{cases}$

$\Rightarrow \begin{cases} -\text{near} = -\text{near} \cdot M_z + M_{zt} \\ \text{far} = -\text{far}\ M_z + M_{zt} \end{cases} \Rightarrow \begin{cases} -\text{near} \cdot \text{far} = -\text{near} \cdot \text{far} \cdot M_z + \text{far} \cdot M_{zt} \\ \text{near} \cdot \text{far} = -\text{near} \cdot \text{far} \cdot M_z + \text{near} \cdot M_{zt} \end{cases}$

$2\text{near} \cdot \text{far} = (\text{near} - \text{far}) \cdot M_{zt}$

① $M_{zt} = \frac{2\text{near} \cdot \text{far}}{\text{near} - \text{far}}$ $\Rightarrow$ $\text{near} = \text{near} \cdot M_z - \frac{2\text{near} \cdot \text{far}}{\text{near} - \text{far}}$

$\Rightarrow \text{near} - \text{far} = (\text{near} - \text{far}) \cdot M_z - 2\text{far}$ $\Rightarrow$ ② $M_z = \frac{\text{near} + \text{far}}{\text{near} - \text{far}}$

①② $\Rightarrow P_z = \frac{\text{near} + \text{far}}{\text{near} - \text{far}} \cdot A_z + \frac{2\text{near} \cdot \text{far}}{\text{near} - \text{far}}$ https://www.desmos.com/calculator/dhsp5blfzg

let $f = \tan(\frac{\pi}{2} - \frac{FOV}{2})$

$\text{rangeInv} = \frac{1}{\text{near} - \text{far}}$

$$\Rightarrow M = \begin{bmatrix} \frac{f}{\text{aspect}} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \text{rangeInv} \cdot (\text{near} + \text{far}) & -1 \\ 0 & 0 & \text{rangeInv} \cdot (2 \cdot \text{near} \cdot \text{far}) & 0 \end{bmatrix}$$